Vojta's Inequality and Rational and Integral Points of Bounded Degree on Curves

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Abstract

Let $C \subset C_1 \times C_2$ be a curve of type (d_1, d_2) in the product of the two curves C_1 and C_2. Let ν be a positive integer. We prove that if a certain inequality involving d_1, d_2, ν, and the genera of the curves C_1, C_2, and C is satisfied, then the set of points $\{P \in C(\bar{k}) \mid [k(P) : k] \leq \nu\}$ is finite for any number field k. We prove a similar result for integral points of bounded degree on C. These results are obtained as consequences of an inequality of Vojta which generalizes the Roth-Wirsing theorem to curves.

1 Introduction

In [16], Vojta proved the following theorem.

Theorem 1 ((Vojta)). *Let C be a nonsingular curve defined over a number field k. Let X be a regular model for C over the ring of integers of k. Let K be the canonical divisor of C, A an ample divisor on C, and D an effective divisor on C without multiple components. Let S be a finite set of places of k. Let ν be a positive integer and let $\epsilon > 0$. Then*

$$m_S(D, P) + h_K(P) \leq d_a(P) + \epsilon h_A(P) + O(1) \tag{1}$$

for all points $P \in C(\bar{k}) \setminus \operatorname{Supp} D$ with $[k(P) : k] \leq \nu$.

Here h_D is a logarithmic height associated to the divisor D, $m_S(D, P)$ is a proximity function, and $d_a(P)$ is the arithmetic discriminant of [15], whose definition we recall below. We refer the reader to [9], [14], and [16] for definitions and properties of heights and proximity functions.

The inequality (1) is a vast generalization of the theorems of Roth and Wirsing. In particular, it implies Faltings' theorem (Mordell's conjecture). As a consequence of (1), Song and Tucker [13] show

Corollary 1 ((Song, Tucker, Vojta)). *Let C and C' be nonsingular curves of genus g and g', respectively, defined over a number field k. Let $\phi : C \to C'$ be a dominant k-morphism. If*

$$g - 1 > (\nu + g' - 1) \deg \phi \tag{2}$$

for some positive integer ν, then the set

$$\{P \in C(\bar{k}) \mid [k(P) : k] \leq \nu \text{ and } k(\phi(P)) = k(P)\} \tag{3}$$

is finite.

Vojta noted the case $C' = \mathbb{P}^1$ of the corollary. Note that the condition $k(\phi(P)) = k(P)$ in Theorem 1 precludes one from deducing a finiteness result on algebraic points with $[k(P) : k] \leq \nu$. Of course, this condition in the theorem is necessary (consider, for example, hyperelliptic curves of genus $g > 3$). If we are given more than one dominant morphism of C to a curve where (2) holds, it is natural to try to prove a finiteness result without the $k(\phi(P)) = k(P)$ condition in (3). Clearly we need the two maps to be independent in some sense. More precisely, we will assume that we are given a birational morphism of C into a product of curves. In addition to rational points, we will study integral points on C.

Let S be a finite set of places of k and let $\mathcal{O}_{k,S}$ be the ring of S-integers of k. Let D be an effective divisor on C. If $D \neq 0$, we call a set $T \subset C(\bar{k}) \backslash \operatorname{Supp} D$ a set of (D, S)-integral points on C if there exists an affine embedding $C \setminus \operatorname{Supp} D \subset \mathbb{A}^m$ such that every point $P \in T$ has S-integral coordinates, i.e., each coordinate of P in $\mathbb{A}^m$ lies in the integral closure of $\mathcal{O}_{k,S}$ in $\bar{k}$. If $D = 0$, then we call any subset of $C(\bar{k})$ a set of (D, S)-integral points. Our main theorem is

Theorem 2. *Let C, C_1, and C_2 be nonsingular curves of genus g, g_1, and g_2, respectively, all defined over a number field k. Let S be a finite set of places k. Let $\phi : C \to C_1 \times C_2$ be a birational morphism. Let π_1 and π_2 be the projections of $C_1 \times C_2$ onto the first and second factors, respectively. Suppose that $\pi_1 \circ \phi$ and $\pi_2 \circ \phi$ are dominant morphisms and let $d_1 = \deg \pi_1 \circ \phi$ and $d_2 = \deg \pi_2 \circ \phi$. Let $D = \sum_{i=1}^{r} P_i$ be an effective divisor on C, defined over k, with $P_1, \ldots, P_r$ distinct points of $C(\bar{k})$. If*

$$2g - 2 + r > \max\{(\nu + g_1 - 1)2d_1, (\nu + g_2 - 1)2d_2, (\nu + 2g_1 - 2)d_1 + (\nu + 2g_2 - 2)d_2\} \tag{4}$$

for some positive integer ν, then any set of (D, S)-integral points

$$T \subset \{P \in C(\bar{k}) \mid [k(P) : k] \leq \nu\}$$

is finite. In particular, if (4) is satisfied with $r = 0$, then the set

$$\{P \in C(\bar{k}) \mid [k(P) : k] \leq \nu\}$$

is finite.

2 Some Examples and Corollaries

We first give two examples which show that the inequality (4) is sharp in the sense that Theorem 2 is false if "$>$" is replaced by "$\geq$" in (4).

Example 1. Let C be a nonsingular curve, defined over a number field k, of bidegree (d_1, d_2) on $C_1 \times C_2 = \mathbb{P}^1 \times \mathbb{P}^1$ with $d_1 \geq d_2 > 0$. Let $P, Q \in \mathbb{P}^1(k)$ be two points above which ϕ_2 is unramified, and let $D = P + Q$. Over sufficiently large number fields k, there are infinitely many k-rational (D, S)-integral points on $\mathbb{P}^1$. Pulling back these integral points by ϕ_2, we obtain infinitely many $(\phi_2^* D, S)$-integral points on $\mathbb{P}^1 \times \mathbb{P}^1$ (of degree $\leq d_2 = \nu$ over k), where $\phi_2^* D$ is a sum of $r = 2d_2$ distinct points. We have $g = (d_1 - 1)(d_2 - 1)$ and we see that equality holds in (4).

Example 2. Let $C_1 \times C_2 = \mathbb{P}^1 \times E$, where E is an elliptic curve defined over a number field k. Let $d_1 > d_2 + 1 > 2$. Let C be a nonsingular curve, defined over a number field k, of type (d_1, d_2) on $\mathbb{P}^1 \times E$ (i.e., $\deg \pi_1|_C = d_1$ and $\deg \pi_2|_C = d_2$). Then by the adjunction formula, $g = g(C) = d_1(d_2 - 1) + 1$. Let $\nu = d_2$ and $r = 0$. Then a simple calculation shows that equality is achieved in (4), but the set $\{P \in C(\bar{k}) \mid [k(P) : k] \leq \nu\}$ is infinite for sufficiently large k since C has a degree $\nu = d_2$ map down to E.

Note that when $C_1 \times C_2 = \mathbb{P}^1 \times \mathbb{P}^1$, the inequality (4) simplifies to

$$2g - 2 + r > \max\{2(\nu - 1)d_1, 2(\nu - 1)d_2\}.$$

As a curve of degree d in $\mathbb{P}^2$ can be mapped birationally onto a curve of bidegree $(d-1, d-1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, we obtain

Corollary 2. *Let $C \subset \mathbb{P}^2$ be a curve, defined over a number field k, of degree d and geometric genus g. Let S be a finite set of places of k. Let $D = \sum_{i=1}^r P_i$ be an effective divisor on C, defined over k, with $P_1, \ldots, P_r$ distinct points of $C(\bar{k})$. If*

$$2g - 2 + r > 2(\nu - 1)(d - 1) \tag{5}$$

for some positive integer ν, then any set of (D, S)-integral points

$$T \subset \{P \in C(\bar{k}) \mid [k(P) : k] \leq \nu\}$$

is finite. In particular, if $g - 1 > (\nu - 1)(d - 1)$ then the set

$$\{P \in C(\bar{k}) \mid [k(P) : k] \leq \nu\}$$

is finite.

By definition, the geometric genus of C is the genus of the normalization of C. For nonsingular plane curves, a better theorem on rational points has been proven by Debarre and Klassen [6] using Falting's theorem on rational points on subvarieties of abelian varieties.

Theorem 3 ((Debarre, Klassen)). *Let $C \subset \mathbb{P}^2$ be a nonsingular curve of degree d, defined over a number field k, that does not admit a map of degree $\leq d-2$ onto a genus one curve (this is automatically satisfied if $d \geq 7$). Then the set*

$$\{P \in C(\bar{k}) \mid [k(P) : k] \leq d-2\}$$

is finite.

Recall that a curve is called hyperelliptic (respectively bielliptic) if it admits a map of degree two onto a curve of geometric genus zero (respectively one). Harris and Silverman [8] have shown (again using Falting's theorem on subvarieties of abelian varieties)

Theorem 4 ((Harris, Silverman)). *Let C be a nonsingular curve defined over a number field k. If C is not hyperelliptic or bielliptic then the set $\{P \in C(\bar{k}) \mid [k(P) : k] \leq 2\}$ is finite.*

A similar theorem is true for degree three rational points (see [1]), but not for degrees four and higher (see [5]). Similarly, for integral points, Corvaja and Zannier [4] have shown

Theorem 5 ((Corvaja, Zannier)). *Let C be a nonsingular curve defined over a number field k. Let S be a finite set of places of k. Let $D = \sum_{i=1}^{r} P_i$ be an effective divisor on C, defined over k, with $P_1, \ldots, P_r$ distinct points of $C(\bar{k})$. Let $T \subset \{P \in C(\bar{k}) \mid [k(P) : k] \leq 2\}$ be a set of (D, S)-integral points. Then*

(a). If $r > 4$ then T is finite.

(b). If $r > 3$ and C is not hyperelliptic then T is finite.

Additionally, in the case C is hyperelliptic and $r = 4$ (where T may be infinite), Corvaja and Zannier show how to parametrize all but finitely many of the quadratic integral points. The proof of Theorem 5 in [4] makes use of an appropriate version of the Schmidt subspace theorem. We now show that Corollary 2 implies a slight improvement to this theorem. Specifically, we show that the inequality in part (b) can be improved to cover the case $r = 3$.

Theorem 6. *Let C be a nonsingular curve defined over a number field k. Let S be a finite set of places of k. Let $D = \sum_{i=1}^{r} P_i$ be an effective divisor on C, defined over k, with $P_1, \ldots, P_r$ distinct points of $C(\bar{k})$. Let $T \subset \{P \in C(\bar{k}) \mid [k(P) : k] \leq 2\}$ be a set of (D, S)-integral points. Then*

(a). If $r > 4$ then T is finite.

(b). If $r > 2$ and C is not hyperelliptic then T is finite.

Proof. By Corollary 2, to prove (a) it suffices to show that any curve C of genus g has a birational plane model of degree $g+2$. Since any divisor of degree $2g+1$ on C is very ample and nonspecial, we obtain an embedding of C as a degree $2g+1$ curve in $\mathbb{P}^{g+1}$. Projecting from the linear span of $g-1$ general points

of C, we obtain a birational map $\phi : C \to \mathbb{P}^2$ with $\deg \phi(C) = g + 2$ (see [2, p. 109]).

Similarly, to prove (b) it suffices to show that if C has genus g and is not hyperelliptic, then C has a birational plane model of degree $g + 1$. Since C is not hyperelliptic, the canonical embedding realizes C as a curve of degree $2g-2$ in $\mathbb{P}^{g-1}$. Projecting from the linear span of $g-3$ general points of C, we obtain a plane curve of degree $g+1$ birational to C. □

As noted in [4], Vojta's conjecture predicts that the inequality in (b) can be improved to $r > 0$. It is unclear to what extent this follows from Theorem 2. For instance, Theorem 2 implies that one may take $r > 0$ in Theorem 6 for any nonsingular bielliptic curve C of type $(a, 2)$, $a > 3$, on $\mathbb{P}^1 \times E$ (of course, by Theorem 4, we need only consider bielliptic curves in (b)).

3 Proofs of Results

Let C be a nonsingular curve defined over a number field k. Let R denote the ring of integers of k and let $B = \operatorname{Spec} R$. Let $\pi : X \to B$ be a regular model for C over R. For every complex embedding $\sigma : k \hookrightarrow \mathbb{C}$ we have a canonical volume form on $C_\sigma = C \times_\sigma \mathbb{C}$ and an associated canonical Green's function g_σ. With this data one can define intersections of Arakelov divisors (see [10]). Let $P \in C(\bar{k})$ and let E_P denote the horizontal prime divisor on X corresponding to P (we will also denote the curve on X corresponding to P by E_P). Let $\omega_{X/B}$ denote the relative dualizing sheaf, with its canonical Arakelov metric [10, Ch. 4]. We then define the arithmetic discriminant $d_a(P)$ by

$$d_a(P) = \frac{(E_P.(\omega_{X/B} + E_P))}{[k(P) : \mathbb{Q}]}.$$

Of course, contrary to the notation, $d_a(P)$ depends on more data than just P. We can also give an alternative formula for $d_a(P)$. Let $L = k(P)$. Then $E_P = \operatorname{Spec} A$, where A is an order of the number field L. Let

$$W_{A/R} = \{b \in L \mid \operatorname{Tr}_{L/k}(bA) \subset R\}$$

be the Dedekind complementary module. It is a fractional ideal of A containing A. For a fractional ideal $\mathfrak{a}$ of A, we define the fractional ideal

$$\mathfrak{a}^{-1} = \{x \in L \mid x\mathfrak{a} \subset A\}.$$

In arbitrary orders, one may not necessarily have $\mathfrak{a}\mathfrak{a}^{-1} = A$. We now define the Dedekind different (of A over R) as

$$\mathcal{D}_{A/R} = W_{A/R}^{-1}.$$

This is an integral ideal of A. For a nice discussion of the relation between the different, discriminant, and conductor of an order, we refer the reader to the

article by Del Corso and Dvornicich [7]. Now define

$$d_{A/R} = \frac{\log[A : \mathcal{D}_{A/R}]}{[L : \mathbb{Q}]}$$

Let S_∞ be the set of archimedean places of k and let $v \in S_\infty$. Let

$$E_v = E_P \times \mathbb{C}_v = \{P_{v,1}, \ldots, P_{v,[L:k]}\}$$

be the points in $C_v = C \times \mathbb{C}_v$ into which E_P splits. By the Arakelov adjunction formula [10, Th. 5.3], we have

$$d_a(P) = d_{A/R} + \frac{1}{[L : \mathbb{Q}]} \sum_{v \in S_\infty} \sum_{i \neq j} N_v \lambda_v(P_{v,i}, P_{v,j}) \tag{6}$$

where $N_v = [k_v : \mathbb{Q}_v]$, and $\lambda_v = \frac{1}{2} g_v$ (with g_v normalized as in [10]). We will use that λ_v is a Weil function for the diagonal Δ_v in $C_v \times C_v$, i.e., if the Cartier divisor Δ_v is locally represented by a function f on the open set U, then there exists a continuous function α on U such that

$$\lambda_v(P) = -\log|f(P)| + \alpha(P)$$

for all $P \in U \setminus \Delta_v$.

Theorem 7. *Let C_1, C_2, and C_3 be nonsingular curves defined over k and let X_1, X_2, and X_3 be regular models over R for the respective curves. Let $\phi : X_3 \to X_1 \times X_2$ be a birational morphism onto its image. Let ϕ_1 and ϕ_2 denote ϕ composed with the projection map of $X_1 \times X_2$ onto the first and second factor, respectively. Let $P \in C_3(\bar{k})$. Then*

$$d_a(P) \leq d_a(\phi_1(P)) + d_a(\phi_2(P)) + O(1). \tag{7}$$

Our strategy is to break up d_a into a finite and infinite part as in (6), and then prove the inequality for each part separately. Since there is an $O(1)$ term, we can clearly ignore the finite set Z of $C(\bar{k})$ on which $\phi_{\bar{k}}$ fails to be invertible. To prove the inequality for the finite part, $d_{A/R}$, of (6), we use the following lemma.

Lemma 1. *Let R be the ring of integers of a number field k. Let A_1 and A_2 be R-orders of the number fields L_1 and L_2, respectively (with some fixed embedding in $\bar{k}$). Let $L_3 = L_1 L_2$ and let $A_3 = A_1 A_2$. If A_1, A_2, and A_3 are Gorenstein rings then*

$$d_{A_3/R} \leq d_{A_1/R} + d_{A_2/R} \tag{8}$$

Proof. As shown in [7], an R-order A is Gorenstein if and only if $\mathcal{D}_{A/R}$ is an invertible ideal of A (see [3] for the many equivalent definitions of a Gorenstein ring). Let A_i' denote the integral closure of A_i in L_i for $i = 1, 2, 3$. For the Gorenstein rings A_1, A_2, and A_3 we have the relations (see [7, Prop. 3])

$$\mathcal{D}_{A_i/R} A_i' = \mathcal{C}_{A_i} \mathcal{D}_{A_i'/R}, \quad i = 1, 2, 3, \tag{9}$$

where

$$\mathcal{C}_{A_i} = \{x \in A_i' \mid xA_i' \subset A_i\}$$

is the conductor of A_i. For an invertible ideal $\mathfrak{a}$ of A_3 (see [7, Th. 1]),

$$[A_3 : \mathfrak{a}] = [{A_3}' : \mathfrak{a}{A_3}'].$$

Now to prove the lemma, it suffices to show that

$$\mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}{A_3}' \subset \mathcal{D}_{A_3/R}{A_3}'.$$

Indeed, this inclusion gives

$$[A_3 : \mathcal{D}_{A_3/R}] = [{A_3}' : \mathcal{D}_{A_3/R}{A_3}'] \leq [{A_3}' : \mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}{A_3}']$$

which is equivalent to (8) as

$$\begin{aligned}[{A_3}' : \mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}{A_3}'] &= [{A_3}' : \mathcal{D}_{A_1/R}{A_3}'][{A_3}' : \mathcal{D}_{A_2/R}{A_3}'] \\ &= [A_1' : \mathcal{D}_{A_1/R}A_1']^{[L_3:L_1]}[{A_2}' : \mathcal{D}_{A_2/R}{A_2}']^{[L_3:L_2]} \\ &= [A_1 : \mathcal{D}_{A_1/R}]^{[L_3:L_1]}[A_2 : \mathcal{D}_{A_2/R}]^{[L_3:L_2]}.\end{aligned}$$

We now show that $\mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}{A_3}' \subset \mathcal{D}_{A_3/R}{A_3}'$. By (9),

$$\mathcal{D}_{A_1/R}\mathcal{D}_{A_2/R}{A_3}' = \mathcal{C}_{A_1}\mathcal{D}_{A_1'/R}\mathcal{C}_{A_2}\mathcal{D}_{{A_2}'/R}{A_3}'$$

and

$$\mathcal{D}_{A_3/R}{A_3}' = \mathcal{C}_{A_3}\mathcal{D}_{{A_3}'/R} = \mathcal{C}_{A_3}\mathcal{D}_{{A_3}'/A_1'}\mathcal{D}_{A_1'/R}.$$

Therefore we need to show $\mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{D}_{{A_2}'/R}{A_3}' \subset \mathcal{C}_{A_3}\mathcal{D}_{{A_3}'/A_1'}$. It is a standard fact that $\mathcal{D}_{{A_2}'/R}$ is generated by elements of the form $f'(\alpha)$, where $\alpha \in {A_2}'$, $k(\alpha) = L_2$, and f is the minimal polynomial of α over k. Let g be the minimal polynomial of α over L_1. Note that $L_1(\alpha) = L_3$ and that $g'(\alpha)$ divides $f'(\alpha)$ in ${A_3}'$. It is easily shown that $g'(\alpha){A_3}' = \mathcal{C}_{A_1'[\alpha]}\mathcal{D}_{{A_3}'/A_1'}$. We have

$$\mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{C}_{A_1'[\alpha]} \subset \mathcal{C}_{A_3}$$

since

$$\mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{C}_{A_1'[\alpha]}{A_3}' \subset \mathcal{C}_{A_1}\mathcal{C}_{A_2}A_1'[\alpha] \subset \mathcal{C}_{A_1}\mathcal{C}_{A_2}A_1'{A_2}' \subset A_1A_2 = A_3.$$

Therefore

$$\mathcal{C}_{A_1}\mathcal{C}_{A_2}f'(\alpha) \subset \mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{C}_{A_1'[\alpha]}\mathcal{D}_{{A_3}'/A_1'} \subset \mathcal{C}_{A_3}\mathcal{D}_{{A_3}'/A_1'}$$

As $D_{{A_2}'/R}$ was generated by the $f'(\alpha)$, we obtain $\mathcal{C}_{A_1}\mathcal{C}_{A_2}\mathcal{D}_{{A_2}'/R}{A_3}' \subset \mathcal{C}_{A_3}\mathcal{D}_{{A_3}'/A_1'}$ as desired. □

Now let $E_P = E_3 = \operatorname{Spec} A_3$ be the prime horizontal divisor corresponding to $P \in C(\bar{k}) \setminus Z$, and let $\phi_1(E_P) = E_1 = \operatorname{Spec} A_1$ and $\phi_2(E_P) = E_2 = \operatorname{Spec} A_2$. Note that A_1 and A_2 are naturally subrings of A_3 (via ϕ_1 and ϕ_2) and $A_3 = A_1 A_2$. Indeed, the closed immersion $\phi : E_P \to X_1 \times X_2$ factors through $E_1 \times E_2$, and therefore the natural map $A_1 \otimes A_2 \to A_3$ is surjective. Since X_1, X_2, and X_3 were assumed regular, E_P, E_1, and E_2 are locally complete intersections (they are Cartier divisors). This implies in particular that A_1, A_2, and A_3 are Gorenstein rings. Therefore, using Lemma 1, we have proved the finite part of the inequality (7), i.e., the inequality (8).

We now consider the archimedean part of (7). With notation as above, let L_1, L_2, and L_3 be the quotient fields of A_1, A_2, and A_3. Let $v \in S_\infty$. Let E_{iv} be the set of points of $E_i \times \mathbb{C}_v$, $i = 1, 2, 3$. Let λ_{Δ_1}, λ_{Δ_2}, and λ_{Δ_3} denote the Weil functions (relative to v) of (6) for C_1, C_2, and C_3, respectively. Here Δ_i is the diagonal of $C_{iv} \times C_{iv}$. Then it suffices to prove

Lemma 2. *In the notation above,*

$$\frac{1}{[L_3 : \mathbb{Q}]} \sum_{\substack{P,Q \in E_{3v} \\ P \neq Q}} \lambda_{\Delta_3}(P, Q) \leq \frac{1}{[L_1 : \mathbb{Q}]} \sum_{\substack{P,Q \in E_{1v} \\ P \neq Q}} \lambda_{\Delta_1}(P, Q) + \frac{1}{[L_2 : \mathbb{Q}]} \sum_{\substack{P,Q \in E_{2v} \\ P \neq Q}} \lambda_{\Delta_2}(P, Q) + O(1).$$

The lemma will follow easily from the following "distribution relation" of Silverman [11, Prop. 6.2(b)] (proved by Silverman in greater generality).

Theorem 8 ((Silverman)). *Let C and C' be nonsingular complex curves. Let $\phi : C \to C'$ be a morphism. Let Δ and Δ' denote the diagonals of $C \times C$ and $C' \times C'$, respectively. Let λ_Δ and $\lambda_{\Delta'}$ be Weil functions associated to Δ and Δ' (under the usual complex absolute value). Then for any $P \in C$ and $q \in C'$ with $\phi(P) \neq q$,*

$$\lambda_{\Delta'}(\phi(P), q) = \sum_{Q \in \phi^{-1}(q)} e_\phi(Q/q) \lambda_\Delta(P, Q) + O(1)$$

where $e_\phi(Q/q)$ is the ramification index of ϕ at Q.

Proof of Lemma 2. Denote by ϕ, ϕ_1, and ϕ_2 the same maps base extended to C_{3v}. Let $P, Q \in E_{3v}$, $P \neq Q$. Since we assumed $P \notin Z$, either $\phi_1(P) \neq \phi_1(Q)$ or $\phi_2(P) \neq \phi_2(Q)$. Note also that the maps $E_{3v} \to E_{1v}$ and $E_{3v} \to E_{2v}$ are $[L_3 : L_1]$-to-1 and $[L_3 : L_2]$-to-1 maps respectively. Thus we obtain (modulo

bounded functions independent of E_{1v}, E_{2v}, and E_{3v})

$$\begin{aligned}
\sum_{\substack{P,Q\in E_{3v}\\ P\neq Q}} \lambda_{\Delta_3}(P,Q) &\leq \sum_{\substack{P,Q\in E_{3v}\\ \phi_1(P)\neq\phi_1(Q)}} \lambda_{\Delta_3}(P,Q) + \sum_{\substack{P,Q\in E_{3v}\\ \phi_2(P)\neq\phi_2(Q)}} \lambda_{\Delta_3}(P,Q)\\
&\leq \sum_{P\in E_{3v}} \sum_{\substack{Q\in\phi_1^{-1}(E_{1v})\\ \phi_1(P)\neq\phi_1(Q)}} \lambda_{\Delta_3}(P,Q) + \sum_{P\in E_{3v}} \sum_{\substack{Q\in\phi_2^{-1}(E_{2v})\\ \phi_2(P)\neq\phi_2(Q)}} \lambda_{\Delta_3}(P,Q)\\
&\leq \sum_{P\in E_{3v}} \sum_{\substack{q\in E_{1v}\\ \phi_1(P)\neq q}} \lambda_{\Delta_1}(\phi_1(P),q) + \sum_{P\in E_{3v}} \sum_{\substack{q\in E_{2v}\\ \phi_1(P)\neq q}} \lambda_{\Delta_2}(\phi_2(P),q)\\
&\leq [L_3:L_1] \sum_{\substack{p,q\in E_{1v}\\ p\neq q}} \lambda_{\Delta_1}(p,q) + [L_3:L_2] \sum_{\substack{p,q\in E_{2v}\\ p\neq q}} \lambda_{\Delta_2}(p,q)
\end{aligned}$$

Dividing everything by $[L_3:\mathbb{Q}]$ gives the lemma. □

Theorem 7 now follows from Lemma 1 and Lemma 2. We now prove Theorem 2 from the introduction. We will need the following estimate of Song and Tucker (see [12] and [13]) for $d_a(P)$ on a curve.

Lemma 3 ((Song, Tucker)). *Let C be a nonsingular curve defined over a number field k with canonical divisor K. Let X be a regular model for C over the ring of integers of k. Let A be an ample divisor on C and let $\epsilon > 0$. Then*

$$d_a(P) \leq h_K(P) + (2[k(P):k]+\epsilon)h_A(P) + O([k(P):k]).$$

Proof of Theorem 2. Let T be as in the hypotheses of Theorem 2 and suppose that the inequality (4) of Theorem 2 is satisfied. Consider the three sets

$$\begin{aligned}
T_1 &= \{P \in T \mid [k(\phi_1(P)):k] = [k(P):k]\}\\
T_2 &= \{P \in T \mid [k(\phi_2(P)):k] = [k(P):k]\}\\
T_3 &= \{P \in T \mid [k(\phi_1(P)):k] < [k(P):k], [k(\phi_2(P)):k] < [k(P):k]\}.
\end{aligned}$$

Clearly $T = T_1 \cup T_2 \cup T_3$. As we assumed $2g-2+r > (\nu+g_1-1)2d_1$ and $2g-2+r > (\nu+g_2-1)2d_2$, it follows from a trivial generalization of Corollary 1 that T_1 and T_2 are finite. So we are reduced to showing that if $2g-2+r > (\nu+2g_1-2)d_1 + (\nu+2g_2-2)d_2$ then T_3 is finite. Let K, K_1, and K_2 denote the canonical divisors of C, C_1, and C_2, respectively. Let h, h_1, and h_2 denote heights associated to some degree one divisor on C, C_1 and C_2, respectively. Using Theorem 1, Theorem 7, and Lemma 3, we get, for any $\epsilon > 0$,

$$\begin{aligned}
m_S(D,P) + h_K(P) &\leq d_a(P) + \epsilon h(P) + O(1)\\
&\leq d_a(\phi_1(P)) + d_a(\phi_2(P)) + \epsilon h(P) + O(1)\\
&\leq h_{K_1}(\phi_1(P)) + (2[k(\phi_1(P)):k]+\epsilon)h_1(\phi_1(P)) +\\
&\qquad h_{K_2}(\phi_2(P)) + (2[k(\phi_2(P)):k]+\epsilon)h_2(\phi_2(P)) + O(1).
\end{aligned}$$

Note that for $P \in T_3$, $[k(\phi_1(P)) : k] \le \nu/2$ and $[k(\phi_2(P)) : k] \le \nu/2$, since $k(\phi_1(P))$ and $k(\phi_2(P))$ are both proper subfields of $k(P)$. Since T is a set of (D,S)-integral points, $m_S(D,P) = h_D(P)+O(1)$ for $P \in T$. Using functoriality of heights and quasi-equivalence of heights associated to numerically equivalent divisors, we obtain, for any $\epsilon > 0$,

$$(2g-2+r)h(P) \le ((\nu+2g_1-2)d_1 + (\nu+2g_2-2)d_2 + \epsilon)h(P) + O(1)$$

for $P \in T_3$. Taking $\epsilon < 1$, since there are only finitely many points of bounded degree and bounded height, we see that if

$$2g-2+r > (\nu+2g_1-2)d_1 + (\nu+2g_2-2)d_2$$

then T_3, and hence T, must be finite. $\square$

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References

[1] Dan Abramovich and Joe Harris, *Abelian varieties and curves in $W_d(C)$*, Compositio Math. **78** (1991), no. 2, 227–238.

[2] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985.

[3] Hyman Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

[4] P. Corvaja and U. Zannier, *On integral points on surfaces*, Ann. of Math. (2) **160** (2004), no. 2, 705–726.

[5] Olivier Debarre and Rachid Fahlaoui, *Abelian varieties in $W_d^r(C)$ and points of bounded degree on algebraic curves*, Compositio Math. **88** (1993), no. 3, 235–249.

[6] Olivier Debarre and Matthew J. Klassen, *Points of low degree on smooth plane curves*, J. Reine Angew. Math. **446** (1994), 81–87.

[7] Ilaria Del Corso and Roberto Dvornicich, *Relations among discriminant, different, and conductor of an order*, J. Algebra **224** (2000), no. 1, 77–90.

[8] Joe Harris and Joe Silverman, *Bielliptic curves and symmetric products*, Proc. Amer. Math. Soc. **112** (1991), no. 2, 347–356.

[9] Serge Lang, *Fundamentals of Diophantine geometry*, Springer-Verlag, New York, 1983.

[10] ———, *Introduction to Arakelov theory*, Springer-Verlag, New York, 1988.

[11] Joseph H. Silverman, *Arithmetic distance functions and height functions in Diophantine geometry*, Math. Ann. **279** (1987), no. 2, 193–216.

[12] Xiangjun Song and Thomas J. Tucker, *Dirichlet's theorem, Vojta's inequality, and Vojta's conjecture*, Compositio Math. **116** (1999), no. 2, 219–238.

[13] ———, *Arithmetic discriminants and morphisms of curves*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 1921–1936 (electronic).

[14] Paul Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Mathematics, vol. 1239, Springer-Verlag, Berlin, 1987.

[15] ———, *Arithmetic discriminants and quadratic points on curves*, Arithmetic algebraic geometry (Texel, 1989), Progr. Math., vol. 89, Birkhäuser Boston, Boston, MA, 1991, pp. 359–376.

[16] ———, *A generalization of theorems of Faltings and Thue-Siegel-Roth-Wirsing*, J. Amer. Math. Soc. **5** (1992), no. 4, 763–804.